DISCOVERY

NEVILLE de MESTRE

Weighty questions

The difference between mass and weight is discussed very early in most physics courses, and those who indulge in mathematical problems involving weights should know the difference. Mass is often defined as the amount of matter in an object. This usually means the sum of the masses of all the atoms that constitute that particular object, but we would then have to define the mass of an atom. We shall not delve too deeply into this, but should note that the mass of an object remains the same no matter where it is in the universe.

Weight, on the other hand, is the force exerted when a mass is acted on by the gravitational attraction of its environment. For objects near the earth's surface, the major gravitational pull is due to the earth, although the sun, moon and planets have a small effect (the moon affects the tides, for example). Now the earth's gravitational force on a mass of 1 kg is GE/R^2, where G is the universal gravitational constant (6.67×10^{-11} MKS units), E is the mass of the earth (5.98×10^{24} kg), and R is the distance of the object from the centre of mass of the earth (6.38×10^6 m at sea-level at the Equator). However, the earth is neither spherical nor smooth, and the value R varies with latitude and with height above or below sea-level. Thus $GE/R^2 = g$ is 9.81 ms^{-2} in Sydney, but only 9.79 ms^{-2} in Mexico City.

For objects on the surface of the moon, the biggest gravitational pull is due to the moon, and the earth has only a small effect. If the radius of the moon is 1.74×10^6 km, and its mass is 7.35×10^{22} kg, ask your students to calculate the value of g on the moon. (Remember that G is true anywhere in the universe).

Thus the weight of an object varies as the object moves around the universe, but it only changes by a small amount within a local environment (your town). Consequently, many people think that 'mass' and 'weight' are equivalent, and they even describe a person's weight in kilograms (the measure of mass) instead of kilograms weight (the measure of force).

So now your students should be able to answer the classic riddle, 'Which weighs more: 50 kg of feathers or 50 kg of lead?' by saying that it depends on whether they are weighed at the same or different places.

Now the weight of an object can be measured by a simple two-pan balance using known masses in one pan of the balance and the object in the other pan, if it can fit. Suppose that the known masses are 1, 2 and 4 kg. Then clearly any object of integer mass up to and including 7 kg can be measured. Fractional masses of 1 kg can also be used, but let us stick with integers at this stage. If we add an 8 kg mass we can now measure up to 15 kg. Note that 5 kg and 10 kg masses can also give us 15 kg, but we can only balance 5, 10 or 15 kg, whereas 1, 2, 4, 8 kg is the same total weight to carry but allows us to balance all the integers from 1 though to 15.

Suppose however, that we want to use only four standard masses but reach as many integer values as possible. Your students should try to solve this problem in stages by using a 'Polya' approach, whereby a simpler (but related) problem is considered first of all. Suppose that we have only two standard masses, then we would have to choose 1 and 2 kg if the masses were to be placed in one pan only. However, if we could place them in either pan we could use 1 and 3 kg and obtain $1 + 0 \equiv 1$, $3 - 1 \equiv 2$, $3 + 0 \equiv 3$, $3 + 1 \equiv 4$ where the first mass mentioned in each equivalence statement is placed in one pan, and the mass after the ± sign is placed in the same pan for '+' and the other pan for '–'.

Next, we need to obtain a combination to produce 5 kg. The largest number that can do this is 9 kg because

$9 - (3 + 1) \equiv 5$

Clearly, we can now obtain all integers up to

$9 + (3 + 1) \equiv 13$

with the three masses 1, 3, 9 kg used in either pan.

It should not take too long for your students to extend the above discussion for the one-pan solution as (1, 2, 4, 8, 16…) and the two-pan solution as (1, 3, 9, 27, 81…). Clearly, if both pans can be used for masses, then the least number needed to weigh up to 121 kg-wt. is five. What is the least number needed for the same problem if only one-pan can be used for the masses?

Here are some more weighing problems.

I have a rock weighing 30 kg-wt. I drop it and it breaks into four pieces. What should be the weight of each piece so that I can balance any object of integer weight up to 30 kg-wt?

The answer to this is not unique.

Another balance problem is to consider placing eight packages into two equal suitcases, which are to be carried by one person. The packages have masses 3, 3.5, 5, 5, 5.5, 6, 6.5 and 7 kg.

Suppose that three people, Alice, Bob and Carol want to know how much they weigh? The available weighing machine is only accurate for loads greater than 100 kg-wt, but each person weighs between 50 and 100 kg-wt. How can they determine their weights?

After the class suggests that they be weighed in pairs, tell them that Alice and Bob together weigh 121 kg-wt, Bob and Carol together weigh 138 kg-wt, while Alice and Carol together weigh 133 kg-wt. See if your students can then write down some equations resulting from this word problem, namely

$a + b = 121$

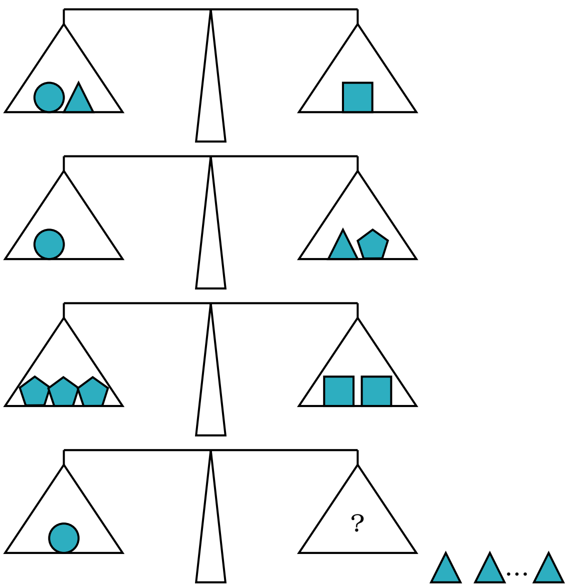
$b + c = 138$

$a + c = 133$

Elimination of c by subtraction yields $b - a = 5$ and hence $a = 58$, $b = 63$, $c = 75$.

Ask your students what they would do if the weighing machine was inaccurate below 130 kg-wt?

Similar problems with balances are often asked using diagrams as follows:



How many triangles would balance the circle on the fourth set of scales?

Next we consider a simple classic logic problem involving 27 balls and a balance. There are no weights and the balls all look alike. However, one is heavier than the other 26. The problem is to find it in three weighings. Again your students could use the 'Polya' technique and find the maximum number of balls for one weighing then two weighings and finally three. For one weighing we can find the heavier one from three balls. For two weighings we can find the heavier one from nine balls, in three groups of three. These preliminary solutions lead us to the final answer.

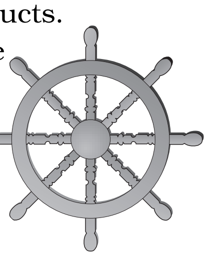
A harder problem is to consider only 12 balls which all look alike, but one has a different weight (i.e., it may be heavier or lighter). How can we find which one it is, and whether it is heavier or lighter in three weighings with the balance? I leave the solution to you.

Consider a consignment of 100 gold ingots each made up of 10 piles of 10 ingots each from different suppliers, with each ingot weighing supposedly 10 kg. It is known that one supplier has supplied ingots that are 10 grams lighter. Without weighing each pile of 10 ingots, how can one weighing determine the light pile? The well-known answer is to mark each supplier's ingots differently from the others, take 1 from the first pile, 2 from the second pile and so on up to 10 from the tenth pile. Then put all these on a balance and weigh them. The weight will be one of 9.99, 9.98, 9.97... 9.90 kg-wt and whichever one it is should be subtracted from 10.00 to give the digit in the second decimal place as the light pile.

Let us now consider the problem of dividing 20 kg of tea into ten 2 kg packets using only a 5 kg-wt and a 9 kg-wt and a minimum of 9 weighings. Your students should be able to find a number of different ways to solve this. One way is to weigh out 14 kg-wt in the first weighing, leaving 6 kg. Then put the 9 kg-wt in one pan and the 5 kg-wt in the other pan and balance with 4 kg of tea from the 6 kg pile. This gives us one 2 kg packet in two weighings. We now use this 2 kg-wt to break up the 4 kg pile (one weighing needed) and the 14 kg pile (six more weighings needed).

Finally, we pose a problem using dishonest scales; one arm of the scales was longer than the other, but this was hidden from the observer by having pans of different weights. If I put three equal circular weights on the long arm, they balance with eight cubic weights on the short arm. Now if I put one cube on the long arm, it balances with six circular weights on the other arm. The circular weights weigh 100 gm-wt each. What is the weight of a cube? You will need to explain about moments before this can be solved, but a see-saw offers an easy explanation and introduces products.

The last problem makes you realise how important it is to check the scales, particularly by putting equal weights in both pans to start off with.



Enlightening discoveries!